Electromagnetic sources beyond common multipoles

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(Received 28 April 2018; published 29 August 2018)

The complete dynamic multipole expansion of electromagnetic sources contains more types of multipole terms than is conventionally perceived. The toroidal multipoles are one of the examples of such contributions that have been widely studied in recent years. Here we inspect more closely the other type of commonly overlooked terms known as the mean-square radii. In particular, we discuss both quantitative and qualitative aspects of the mean-square radii and provide a general geometrical framework for their visualization. We also consider the role of the mean-square radii in expanding the family of nontrivial nonradiating electromagnetic sources.

DOI: 10.1103/PhysRevA.98.023858

I. INTRODUCTION

Multipole expansion is one of the main analytical instruments of modern theoretical physics. In electrodynamics it allows one to describe electromagnetic properties of a charge-current excitation of any spatial complexity and is routinely used for simplifying the analysis of a wide range of electromagnetic systems—from elementary particles and nuclei to neutron stars and black holes. The dynamic multipole expansion is commonly derived as a series of terms of two different types, the so-called electric and magnetic multipoles, which correspond to elementary sources of electromagnetic radiation formed by oscillating charges and circulating currents, respectively. It was indicated by several groups [1–3] and recently confirmed experimentally [4] that the common expansion is missing toroidal multipoles—a third independent family of elementary sources, which is usually overlooked in the course of the expansion but plays an important role in metamaterial, plasmonic, and nanophotonic systems [5]. It is little known, however, that every single multipole in the expansion (be it an electric, a magnetic, or a toroidal one) gives rise to a subset of additional higher-order terms, that are referred to as the *mean-square radii* (or *MSRs*), of the respective multipole and without which the standard multipole expansion cannot be complete [2]. In this paper we determine and visualize charge-current distributions corresponding to the MSRs and identify realistic electromagnetic systems capable of supporting such excitations. We also show that the MSRs give rise to three distinct groups of nonradiating (NR) electromagnetic systems that do not involve the interference of electrical and toroidal multipoles (as in a dynamic anapole [1]).

We begin with the charge distribution that can serve as a faithful representation of the most elementary static MSR and will later help us to construct and visualize explicit examples of other MSRs (see Fig. 1). Here a point charge $+q$ is placed in the center of a sphere bearing the total charge of $-q$. It is easy to see that all the standard multipoles in this system are absent. Indeed, since there is no current, both magnetic and toroidal multipoles vanish. Due to the spherical symmetry all the electric multipoles (dipole, quadrupole, etc.) also vanish. Correspondingly, the standard multipole expansion of the system at hand (and, in general, any spherically symmetric electrostatic system) is reduced to a single quantity—its total electric charge. Only the total charge affects the electric field outside a spherically symmetric system, and when it is zero, the external electric field is absent. At the same time, the system that looks trivial on the outside may still remain nontrivial internally, and the standard multipoles fail to capture that. In our case the total charge Q is zero by construction, which is expressed mathematically as

$$Q = \int d\boldsymbol{r}\, \rho(\boldsymbol{r}) = 0, \tag{1}$$

with $\rho(\boldsymbol{r})$ being the charge density of the system. Although the result of integration is zero, the charge density clearly does not vanish everywhere (see Fig. 1). In fact, it could have any radial distribution so long as it preserved the spherical symmetry (and total charge). To "encode" the details of this radial profile the following set of quantities can be used (which have been derived by introducing weight factors in the expression for the total charge):

$$Q^{(n)} = \int d\boldsymbol{r}\, r^{2n} \rho(\boldsymbol{r}). \tag{2}$$

The form of Eq. (2) implies that $Q^{(n)}$ is the nth-order mean-square radius of an electric charge. A simple computation shows that in our case $Q^{(n)} = -qR^{2n}$, where n is a positive integer and R is the sphere's radius. Clearly, in the limit of vanishing R the internal structure of the above system can be captured simply by the MSR of the first order or, in other words, Fig. 1 is a graphical representation of the first MSR of an electric charge.

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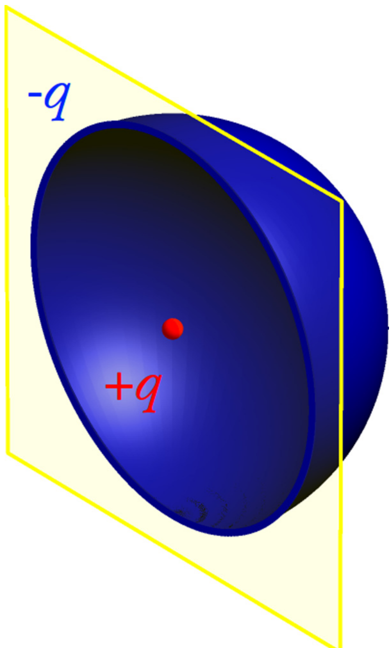

FIG. 1. A model of the static first-order mean-square radius of a point charge: a point charge ($+q$) placed in the center of an oppositely charged sphere bearing the same total charge ($-q$). Only half of the sphere is shown explicitly.

Quite generally, each term in the standard multipole expansion captures only a certain angular projection of the charge-current distribution and leaves its radial smearing unaccounted for. To define an electromagnetic source fully, the standard multipole expansion must be supplemented with a series of MSRs for every multipole term there is. Although the choice of MSRs for encoding the radial profile of a charge distribution may seem arbitrary for static sources (since static MSRs do not affect the external fields in any way), in the dynamic multipole expansion, MSRs emerge naturally because most of them contribute to the electromagnetic radiation just as their parent multipoles do [2,3] (see also Sec. VI). We now proceed to the description and analysis of general time-dependent charge-current sources and their multipole expansion. We will not denote dependence on time explicitly to lighten the notation.

II. MATHEMATICAL REPRESENTATION OF MEAN-SQUARE RADII

The nth MSR of the electric multipole Q_{lm} is defined by

$$Q_{lm}^{(n)} = C_l^n \sqrt{\frac{4\pi}{2l+1}} \int d\boldsymbol{r}\, r^{l+2n} Y_{lm}^*(\hat{\boldsymbol{r}}) \rho(\boldsymbol{r}), \tag{3}$$

where $C_l^n = \frac{2^{-n}(2l+1)!!}{(2l+2n+1)!!}$. For $n=0$ this expression reduces to the standard definition of the multipole moment itself, i.e., $Q_{lm}^{(0)} \equiv Q_{lm}$. For $n \neq 0$ the only difference (apart from the normalization factor C_l^n) is the additional weight factor r^{2n} in the integrand. The MSRs of the magnetic $M_{lm}^{(n)}$ and toroidal $T_{lm}^{(n)}$ multipoles are defined in exactly the same way.[1] Explicit formulas are relegated to Appendix A.

For each electric multipole there is specific (singular) charge density which gives rise only to this multipole and no other. For example, the total charge (i.e., monopole) corresponds to $\rho_q(\boldsymbol{r}) = q\delta(\boldsymbol{r})$, whereas the electric dipole corresponds to $\rho_{\boldsymbol{d}}(\boldsymbol{r}) = -(\boldsymbol{d}\cdot\nabla)\delta(\boldsymbol{r})$. In general, the charge density corresponding to the lmth multipole can be written as [2]

$$\rho_{lm}(\boldsymbol{r}) = \widehat{D}_{lm}(\nabla)\delta(\boldsymbol{r}). \tag{4}$$

Here $\widehat{D}_{lm}$ is a differential operator whose explicit form is not important for our purposes [clearly, $\widehat{D}_{lm}$ is a constant for a monopole $\widehat{D}_q = q$, whereas for a dipole it is $\widehat{D}_{\boldsymbol{d}} = -(\boldsymbol{d}\cdot\nabla)$].

Likewise, there are specific charge-densities $\rho_{lm}^{(n)}(\boldsymbol{r})$ representing the MSRs of the electric multipoles. They are derived by simply applying the Laplace operator $\Delta = \nabla_x^2 + \nabla_y^2 + \nabla_z^2$ to the respective charge density, namely,

$$\rho_{lm}^{(n)}(\boldsymbol{r}) = q_{lm}^{(n)} \Delta^n \rho_{lm}(\boldsymbol{r}). \tag{5}$$

Constants $q_{lm}^{(n)}$ determine the exact values of the corresponding MSRs.[2]

To illustrate formula (5) let us take a closer look at the charge density in Fig. 1,

$$\rho(\boldsymbol{r}) = q\delta(\boldsymbol{r}) - \sigma \int d\boldsymbol{n}\, \delta(\boldsymbol{r} - R\boldsymbol{n}). \tag{6}$$

The integral here runs over the unit vector $\boldsymbol{n}$, which parametrizes the surface of the sphere $\int d\boldsymbol{n} = 4\pi$, whereas $\sigma = q/4\pi$ is the surface charge density. Expanding this expression in the limit of small R to the leading nonvanishing order, and making use of the following relations $\int d\boldsymbol{n}\, n_i = 0$, $\int d\boldsymbol{n}\, n_i n_j = 4\pi\delta_{ij}/3$ yields

$$\begin{aligned}\rho(\boldsymbol{r}) &= q\delta(\boldsymbol{r}) - \sigma \int d\boldsymbol{n}\, [\delta(\boldsymbol{r}) + Rn_i\nabla_i\delta(\boldsymbol{r}) \\ &\quad + \frac{R^2}{2} n_i n_j \nabla_i\nabla_j\delta(\boldsymbol{r})] + O(R^3) \\ &= -\frac{qR^2}{6}\Delta\delta(\boldsymbol{r}) + O(R^3).\end{aligned} \tag{7}$$

We see that, to the leading order in R, this charge density is given by $\rho(\boldsymbol{r}) \propto \Delta\delta(\boldsymbol{r})$ and, hence, indeed corresponds to the first MSR of the electric charge.

The formalism outlined above can be generalized to currents and the respective multipole families. The result is straightforward. If the current-density $\boldsymbol{j}_{lm}$ represents the lmth multipole (magnetic M_{lm} or toroidal T_{lm}), then the corresponding nth MSR is generated by (normalization omitted for simplicity)

$$\boldsymbol{j}_{lm}^{(n)}(\boldsymbol{r}) \propto \Delta^n \boldsymbol{j}_{lm}(\boldsymbol{r}). \tag{8}$$

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[1]Quantities $Q_{lm}^{(n)}$, $M_{lm}^{(n)}$, and $T_{lm}^{(n)}$ without the normalization factor C_l^n are sometimes denoted by $\overline{r_{lm}^{(2n)}}$, $\overline{\rho_{lm}^{(2n)}}$, and $\overline{R_{lm}^{(2n)}}$ [3]. Note that for simplicity we have defined $Q^{(n)}$ in Eq. (2) without proper normalization.

[2]By substituting density (5) in definition (3) one discovers that $Q_{lm}^{(n)} = q_{lm}^{(n)} C_l^n \frac{(2n+l+1)!}{(l+1)!}$.

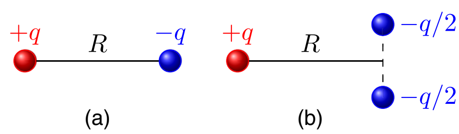


FIG. 2. (a) An electric dipole as a pair of opposite point charges and (b) its alternative representation involving three charges.

III. VISUALIZATION OF FIRST-ORDER MEAN-SQUARE RADII

Having formally associated charge-current density with MSRs of the multipoles of various orders, our next goal is to find a way of visualizing first MSRs of electric, magnetic, and toroidal dipoles. But first, it might be instructive to recall how one arrives at the graphical interpretation of a dipole of the most simple form, i.e., electric dipole.

An electric dipole is usually pictured as a pair of opposite charges, see Fig. 2(a). It should be stressed that the configuration in Fig. 2(a) is not an ideal (pure) dipole. It also embodies an electric quadrupole as well as other higher-order multipoles. The corresponding charge density is given by

$$\rho(\boldsymbol{r}) = q\delta(\boldsymbol{r}) - q\delta(\boldsymbol{r}-\boldsymbol{R}) = q(\boldsymbol{R}\cdot\nabla)\delta(\boldsymbol{r}) + O(R^2). \quad (9)$$

It reduces to the pure dipole density strictly in the limit of vanishing R. This example illustrates a general problem: One cannot accurately represent multipoles (which are pointlike sources) with a finite resolution figure. Any charge-current configuration that one can draw for a given multipole will always contain an admixture of higher-order multipoles. Only in the limit of vanishing size of the configuration, one of the multipoles will become dominant, whereas all others can be neglected. Another side of this problem is that one can draw many charge-current configurations representing a given multipole. For instance, a more complex system in Fig. 2(b) can also serve as an embodiment of the electric dipole, although it is distinct from the one shown in Fig. 2(a). Although the electric dipole representation in Fig. 2(a) appears as simple as it can be, for higher-order multipoles there may not be a single optimal choice (and we will encounter explicit examples of that later). Given the above reservations we can say that the configuration in Fig. 1 faithfully portrays the first MSR of an electric charge and we may now proceed to rendering first-order MSRs of various dipoles.

There is a rather simple way of performing this. Any charge-density $\rho(\boldsymbol{r})$ can be thought of as an assembly of point charges. The first MSR of $\rho(\boldsymbol{r})$ then will be generated by first MSRs of the point charges distributed in exactly the same manner.

Let us prove the validity of this approach. The statement that $\rho(\boldsymbol{r})$ can be represented as an assembly of point charges is formally written as

$$\rho(\boldsymbol{r}) = \int d\boldsymbol{r}'\rho(\boldsymbol{r}')\delta(\boldsymbol{r}-\boldsymbol{r}'). \quad (10)$$

Then, the first mean-square radius of this density $\Delta\rho(\boldsymbol{r})$ can be written as

$$\Delta\rho(\boldsymbol{r}) = \int d\boldsymbol{r}'\rho(\boldsymbol{r}')\Delta\delta(\boldsymbol{r}-\boldsymbol{r}'). \quad (11)$$

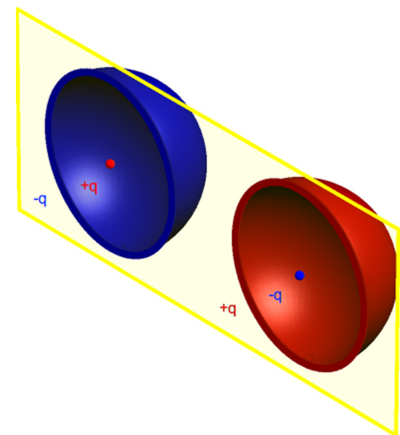


FIG. 3. A model of the static first-order mean-square radius of an electric dipole.

So indeed, in order to reproduce $\Delta\rho(\boldsymbol{r})$ one needs to distribute $\Delta\delta(\boldsymbol{r})$ (which is nothing else but the first MSR of a point charge) with the same density profile $\rho(\boldsymbol{r})$ that originally described the distribution of point charges.

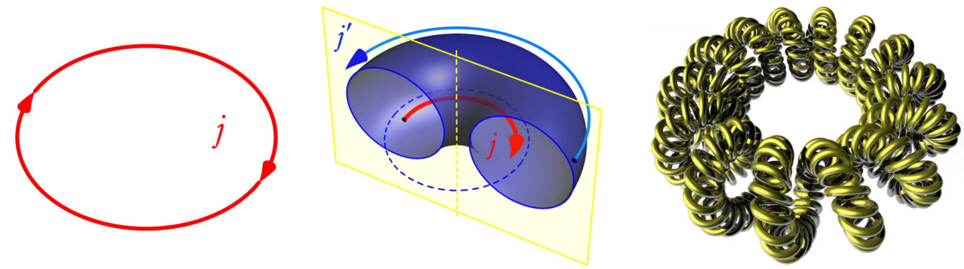
Using the above recipe, we can immediately draw the first-order MSR of an electric dipole as a pair of first MSRs of its point charges (see Fig. 3). Note that, in general, while replacing each point charge with its first MSR, one has to take into account a possible overlap between the charged spheres of adjacent MSRs. Although the resulting picture may not be the simplest representation of the sought after MSR, it will provide a good starting point (as in the case of magnetic and toroidal dipoles below).

A magnetic dipole is usually visualized as a current loop, see Fig. 4(a). We start building its first MSR by replacing every point charge in the current loop with the respective mean-square radius construct. This will result in a complex current source where the current loop threads through the middle of a torus that sustains current in its volume in the opposite direction. Fortunately, it is possible to simplify this picture by replacing the volumetric current inside the torus with a current flowing on its surface, which we denote as $\boldsymbol{j}'$. The simplest replacement prescription requires that the local density of surface current $\boldsymbol{j}'$ decreases linearly with the distance from the torus axis and that the torus major radius will need to become slightly larger than the radius of the current loop, see Fig. 4(b).[3] The exact relation is

$$R^2 = R'^2 - \rho'^2/2, \quad (12)$$

where R is the radius of the current loop whereas R' and ρ' are the major and minor radii respectively, of the torus [radius ρ is not to be confused with the charge density $\rho(\boldsymbol{r})$]. The calculations detailing the origin of this relation can be

[3]Heuristically, overlaps between the imaginary charged spheres circling along the loop are greater towards the center of the loop, so the resulting volumetric current will be radially inhomogeneous. It is this inhomogeneity that is accounted for by the increase in the torus major radius R' and by the variation of surface current density $\boldsymbol{j}'$.



(a) (b) (c)

FIG. 4. Visualizing the first-order mean-square radius of a magnetic dipole. (a) Current distribution representing a magnetic dipole (current circulates in a loop). (b) A cross section of the current distribution representing the first-order MSR of the magnetic dipole. Currents are confined to the red loop ($\boldsymbol{j}$) and the surface of the (imaginary) blue torus ($\boldsymbol{j}'$) and circulate in the opposite directions. The dashed blue circle shows the centerline of the torus. The dashed yellow line is the axis of the torus. (c) A current-carrying supertoroidal wire coil of the third order. In the limit of tight winding, when helicity of the coil vanishes, it represents a source of the first-order MSR of a magnetic dipole.

found in Appendix B. We should stress that fine-tuning of the MSR geometry according to Eq. (12) is only needed to completely eliminate all usual multipoles of the same order as the constructed MSR. As such, this is rather an academic exercise. For any similar configuration (e.g., when the major radius of the torus coincides with the radius of the inner current loop) the first-order MSR of the magnetic dipole would still dominate the contributions of the usual multipoles and, hence, could not be neglected.

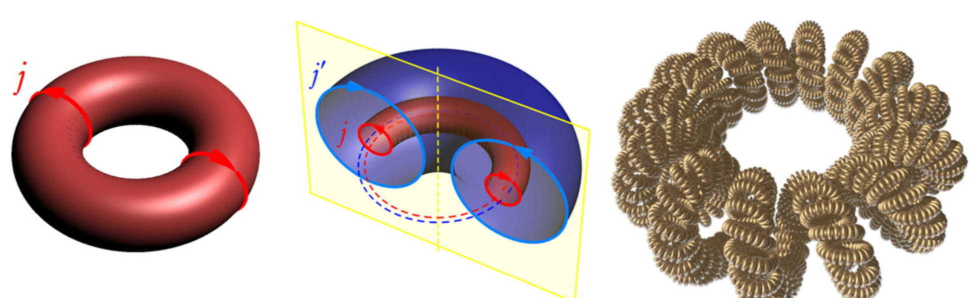
Finally, let us draw the first MSR of a toroidal dipole. The dipole itself is usually represented by poloidal currents flowing on the surface of an imaginary torus, see Fig. 5(a). Replacing charges in these currents with their first MSRs will render a larger thick torus, which encloses the original torus with poloidal currents and contains a volumetric current circulating in the opposite direction. As in the previous case, it is possible to simplify this picture by replacing volumetric currents with surface ones. Although, in general, the described procedure yields three nested tori, it is always possible to shrink the innermost torus to a ring so that the contribution of its surface currents vanishes. The result is two nested tori with surface currents circulating in the opposite directions, see Fig. 5(b). The price to pay for this simplification is a mismatch between surface current densities and major radii of the two tori.[4] The exact relation between the radii, which makes Fig. 5(b) a faithful representation of the first MSR of the toroidal dipole, has the form

$$R^2 - \rho^2/4 = R'^2 - \rho'^2/4, \tag{13}$$

with R and ρ being the major and minor radii, respectively, of the inner torus (and R' and ρ' for the outer torus). This formula is obtained in Appendix C.

To briefly summarize this section, just as the first-order MSR of a point charge is obtained by placing the charge inside an oppositely charged sphere, the first-order MSR of both magnetic and toroidal dipoles can be obtained by placing the corresponding current distribution inside a torus with surface currents circulating in the opposite direction.

[4]Note that the local surface density of currents $\boldsymbol{j}$ and $\boldsymbol{j}'$ is forced to decrease linearly away from the symmetry axis by the current conservation.



(a) (b) (c)

FIG. 5. Visualizing the first-order mean-square radius of a toroidal dipole. (a) The current distribution representing a toroidal dipole (currents circulate on a surface of an imaginary torus along its meridians). (b) A cross section of the current distribution representing the first-order MSR of the toroidal dipole. Currents $\boldsymbol{j}$ and $\boldsymbol{j}'$ are confined to the surfaces of nested (imaginary) red and blue tori and circulate along their meridians in the opposite directions. The dashed blue and red circles show the centerlines of blue and red tori, respectively. The dashed yellow line is the axis of both tori. (c) A current-carrying supertoroidal wire coil of the fourth order. In the limit of tight winding, when the helicity of the coil vanishes, it represents a source of the first-order MSR of a toroidal dipole.

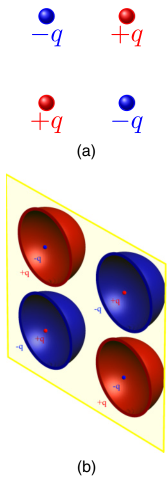


FIG. 6. (a) An electric quadrupole as a set of four point charges and (b) its first-order MSR given by a set of four first MSRs of the point charges.

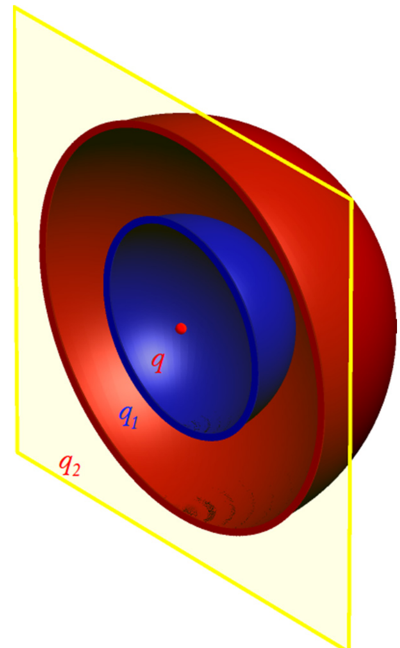


FIG. 7. A model of the static second-order mean-square radius of a point charge.

IV. FURTHER EXAMPLES OF MEAN-SQUARE RADII

In the previous section we gave a general prescription for constructing a first-order MSR of any charge-current configuration and illustrated it using the examples of electric, magnetic, and toroidal dipoles. Here we will show how the procedure can be generalized for higher-order multipoles. An electric quadrupole is usually pictured as four pairwise opposite charges, see Figs. 6(b) and 6(a). To obtain its first-order MSR one just needs to replace each constituent point charge with the corresponding first MSR, see Fig. 6(b). Alternatively, one may view an electric quadrupole as a combination of two oppositely directed electric dipoles. The first MSR of the electric quadrupole is then given simply by a combination of two first MSRs of these electric dipoles. This approach equally applies to magnetic and toroidal quadrupoles, which may be viewed as pairs of the corresponding dipoles. Understanding of how to construct the first MSRs of dipoles and quadrupoles allows one to effortlessly imagine their forms in the case of other higher-order multipoles.

Another direction of generalization is constructing the second- (and higher-) order MSRs. One can show that the configuration depicted in Fig. 7 corresponds to the second MSR of an electric charge if the following two conditions are met

$$q + q_1 + q_2 = 0, \tag{14}$$

$$q_1 R_1^2 + q_2 R_2^2 = 0. \tag{15}$$

Here R_1 and R_2 are the radii of the spheres bearing charges q_1 and q_2, respectively. Equation (14) simply means that the total charge is zero whereas Eq. (15) ensures that the first MSR is zero. In particular, it implies that charges q_1 and q_2 must have opposite signs. In other words, to obtain the second-order MSR of a point charge, the latter needs to be screened not by one but by two charged spheres, and the charges they bear must have opposite signs.

The procedure for constructing the second-order MSR of a magnetic (and toroidal) dipole is also straightforward. One needs to place the corresponding first-order MSR fully inside another (larger) imaginary torus where the surface currents are reversed with respect to those on the outer torus of the first MSR. If the current densities and geometrical parameters of the tori are chosen such that the dipole moment and its first-order MSR are zero, then the resulting configuration will represent the second-order MSR of the dipole.

V. PHYSICAL REALIZATIONS OF MEAN-SQUARE RADII

Let us now identify possible physical realizations of radiating MSR sources. Although this may seem a very daunting task (given the extreme three-dimensional complexity of the underlying current configurations), the realization of such sources is fairly trivial if one recalls the so-called supertoroidal currents. They represent a curious class of fractal current configurations where each iteration replaces every current loop from the previous iteration with a toroidal solenoid formed by smaller loops [6,7]. The supertoroidal current density of nth order with its symmetry axis oriented along the z axis is given by the following formula (the normalization constant is omitted):

$$\boldsymbol{j}_n(\boldsymbol{r}) = \mathrm{rot}^n\, \hat{\boldsymbol{z}}\delta(\boldsymbol{r}). \tag{16}$$

It is easy to see that for $n = 1$ and $n = 2$ the above current density corresponds to magnetic and toroidal dipoles, respectively: $\boldsymbol{j}_0(\boldsymbol{r}) \propto \boldsymbol{j}_\mu = \mathrm{rot}\, \hat{\boldsymbol{z}}\delta(\boldsymbol{r})$, $\boldsymbol{j}_1(\boldsymbol{r}) \propto \boldsymbol{j}_\tau = \mathrm{rot}^2\, \hat{\boldsymbol{z}}\delta(\boldsymbol{r})$. Although in practice the magnetic dipole is produced by a current loop, the toroidal dipole can be generated by currents flowing through a wire solenoid bent into a torus, i.e., toroidal solenoid (see Fig. 8).

The current configuration corresponding to $n = 3$ can be realized with a wire solenoid wrapped around a torus, see Fig. 4(c). Intriguingly, in the limit of vanishing helicity and size of the windings, both magnetic and toroidal (and, naturally,

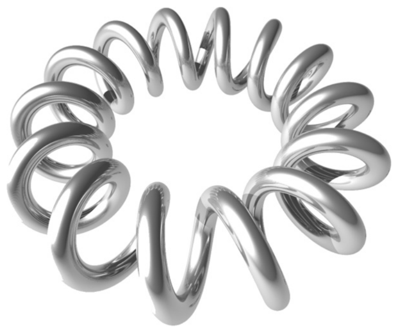


FIG. 8. A current-carrying supertoroidal wire coil of the second order (toroidal solenoid). In the limit of tight winding, when helicity of the coil vanishes, it represents a source of a toroidal dipole.

electric) dipole moments of such a source are zero, yet the supertoroidal current will give off electromagnetic radiation of the dipole type [6,7]. What defines its radiation properties then? The easiest way to find this out is to employ the following transformation for the underlying current density:

$$\begin{aligned} \boldsymbol{j}_3 &= \mathrm{rot}^3\, \hat{z}\delta(\boldsymbol{r}) = (\nabla\ \mathrm{div} - \Delta)\,\mathrm{rot}\ \hat{z}\delta(\boldsymbol{r}) \\ &= -\Delta\ \mathrm{rot}\,\hat{z}\delta(\boldsymbol{r}). \end{aligned} \tag{17}$$

It is now clear that the supertoroidal current of the third order is nothing else but the first MSR of the magnetic dipole. Also, one can also deduce directly from Fig. 4(c) that in the limit of small overlapping loops the current distribution imposed by the supertoroidal coil will transform into the current distribution in Fig. 4(b), which visualizes exactly the first MSR of a magnetic dipole.

One can show in a similar way that the current in a supertoroidal coil of the fourth-order $\boldsymbol{j}_4 = -\Delta\ \mathrm{rot}^2\, \hat{z}\delta(\boldsymbol{r})$ yields the first MSR of a toroidal dipole [Fig. 5(c)], whereas the fifth-order current coil $j_5 = \Delta^2\,\mathrm{rot}\ \hat{z}\delta(\boldsymbol{r})$ corresponds to the second MSR of a magnetic dipole [Fig. 9]. In general, $\boldsymbol{j}_n$ in Eq. (16) will generate the $(n-2)$th MSR of a magnetic dipole for even n and the $(n-3)$th MSR of a toroidal dipole for odd n.

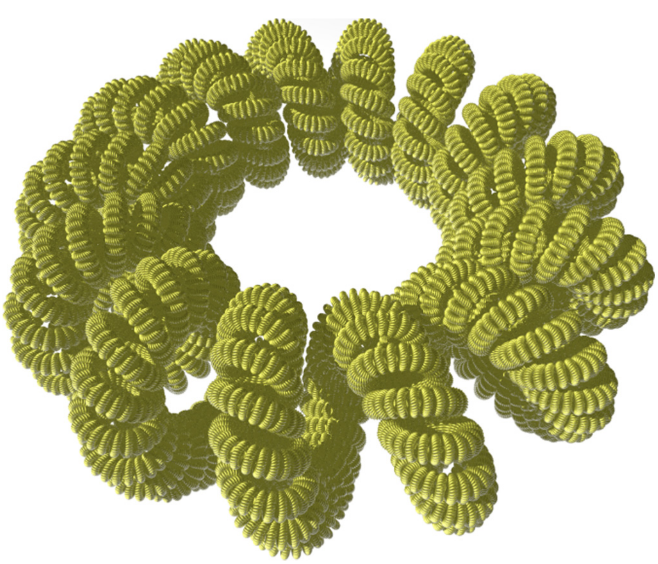


FIG. 9. A current-carrying supertoroidal wire coil of the fifth order. In the limit of tight winding, when helicity of the coil vanishes, it represents a source of the second-order MSR of a magnetic dipole.

VI. ELECTROMAGNETIC PROPERTIES OF MEAN-SQUARE RADII

Every multipole moment of a charge-current source contributes to the radiated electromagnetic field. A crucial yet often underestimated fact is that different multipoles can provide the same contributions to the far-field radiation. The most celebrated illustration of this fact is the radiation patterns of toroidal and electric dipoles, which are identical. The same holds for the higher-order multipoles of electric and toroidal families: The radiation of a toroidal quadrupole is indistinguishable from the radiation of an electric quadrupole, etc. Mean-square radii further expand the library of examples, illustrating the above fact. Indeed, the full radiation intensity of a charge-current source described in terms of its multipole moments has the following form [3]:

$$\begin{aligned} I = c\sum_{l=1}^{\infty}\sum_{m=-l}^{l}&\frac{(l+1)}{l(2l-1)!!(2l+1)!!}k^{2l+2} \\ &\times\left(\left|Q_{lm}^{(0)} + ik\sum_{n=0}^{\infty}\frac{(-1)^n k^{2n}}{n!}T_{lm}^{(n)}\right|^2\right. \\ &\left.+\left|\sum_{n=0}^{\infty}\frac{(-1)^n k^{2n}}{n!}M_{lm}^{(n)}\right|^2\right). \end{aligned} \tag{18}$$

We assumed harmonic time dependence with frequency ω and denoted $k = \omega/c$. Formula (18) concisely summarizes many important features of the *complete* multipole expansion. For example, it shows that the radiation of a toroidal multipole $T_{lm}^{(0)}$ can cancel the radiation of a charge multipole $Q_{lm}^{(0)}$ if the relation $Q_{lm}^{(0)} + ikT_{lm}^{(0)} = 0$ is satisfied. At $l = 1$ it gives the familiar anapole condition $\boldsymbol{d} + ik\boldsymbol{\tau} = 0$ [6].

Moreover, the formula implies that MSRs of magnetic and toroidal multipoles not only radiate, but also have exactly the same radiation pattern as their parent multipoles. For example, the fields radiated by the first MSR of a magnetic dipole are indistinguishable from those radiated by the dipole itself. This, of course, assumes that the magnitudes and phases of the corresponding moments are adjusted properly. Note also that the fields of MSRs scale differently with k and so for a source with fixed geometry the relative contributions of different MSRs will change as the wavelength changes. However, in the most common regime of electrodynamics (i.e., in the long-wavelength limit) all higher-order multipoles and MSRs generically become negligible. Tables I and II place the first few MSRs in the hierarchy of the *complete* multipole expansion.

Note that Eq. (18) is missing MSRs of charge multipoles since they do not radiate. This can be appreciated by revisiting the first static MSR of a point charge shown in Fig. 1. For this charge configuration to remain the first mean-square radius in the dynamic case the oscillations of the shell must preserve spherical symmetry of the configuration. This will be

TABLE I. Multipole terms (up to order 6) that make up *the charge-current distribution* in an electromagnetic source, organized by their origin. Notation $X_m^{(n)}$ means the nth MSR of the mth multipole type X. For example, M_2 is the magnetic dipole whereas $Q_8^{(1)}$ is the first MSR of the electric octupole.

Expansion order	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
Electric type	Q_1	Q_2	Q_4	Q_8	Q_{16}	Q_{32}
			$Q_1^{(1)}$	$Q_2^{(1)}$	$Q_4^{(1)}$	$Q_8^{(1)}$
					$Q_1^{(2)}$	$Q_2^{(2)}$
Magnetic type			M_2	M_4	M_8	M_{16}
					$M_2^{(1)}$	$M_4^{(1)}$
Toroidal type				T_2	T_4	T_8
						$T_2^{(1)}$

possible only for radial oscillations, which naturally produce no electromagnetic waves.

VII. NONRADIATING SOURCES

A NR source is a nontrivial charge-current configuration, which creates no electromagnetic fields outside the volume it physically occupies. Perhaps, the most well-known example is that of the elementary dynamic anapole [5], which consists of electric $\boldsymbol{d}$ and toroidal dipoles $\boldsymbol{\tau}$ whose complex amplitudes are related to each other as $\boldsymbol{d} = -ik\boldsymbol{\tau}$. The multipole expansion of nonradiating sources contains some peculiarities, which we would like to outline with the help of Eq. (18). The necessary and sufficient conditions for the absence of radiated electromagnetic fields are met when the total radiation intensity equals zero [8]. The trivial solution for $I = 0$ is obtained when all multipole moments and their MSRs vanish, implying that space is empty. Nontrivial solutions correspond to destructive interference between different multipole modes. It seems tenable to introduce four types of nontrivial nonradiating sources based on the type of destructive interference involved.

(1) *Anapole type*. It arises from the interference between charge (i.e., electrical) and toroidal multipoles. A familiar example is the anapole.

(2) *Electric type*. This type comes into play when the multipole expansion contains only the MSRs of an electric charge.

TABLE II. Multipole terms (up to order 6) that *contribute to radiation* from an electromagnetic source, organized by the radiation type. Notation $X_m^{(n)}$ means the nth MSR of the mth multipole type X. For example, T_2 is the toroidal dipole whereas $M_4^{(1)}$ is the first MSR of the magnetic quadrupole.

Degree of spherical harmonic (radiation pattern)	$l=1$	$l=2$	$l=3$	$l=4$	$l=5$
Electric type	Q_2	Q_4	Q_8	Q_{16}	Q_{32}
	T_2	T_4	T_8		
	$T_2^{(1)}$				
Magnetic type	M_2	M_4	M_8	M_{16}	
	$M_2^{(1)}$	$M_4^{(1)}$			

Since they do not contribute to radiation, the corresponding source can be regarded as NR.

(3) *Magnetic type*. It arises from the interference between magnetic multipoles and their own MSRs. The condition for destructive interference is met when the second squared term in Eq. (18) vanishes. For example, for the lowest-order NR source of this type, which is formed by a magnetic dipole and its first MSR, the nonradiating condition is $\boldsymbol{\mu} = k^2\boldsymbol{\mu}^{(1)}$.[5]

(4) *Toroidal type*. Toroidal multipoles can interfere with their own MSRs (just as magnetic multipoles do) and form NR sources even in the absence of electric multipoles. The lowest-order NR source of this type is formed by a toroidal dipole and its first MSR when $\boldsymbol{\tau} = k^2\boldsymbol{\tau}^{(1)}$.

Consequently, any combination of the NR sources of the above types will also lead to a NR source. An important remark is in order. Recall that the multipole moments generally depend on the choice of the coordinate origin. Only the leading multipole moment (or moments if there are several of the same level) is invariant upon coordinate shift. Since in a NR source the interference occurs between a lower-order multipole term and a higher-order one (which is not invariant with respect to coordinate shift), the classification above is to a certain extent inaccurate. Take, for example, an NR source of electric type formed only by charge MSRs. A change in the origin (and for a real source no point can be preferred as the origin) will inevitably introduce an admixture of other multipole modes. The source will remain nonradiative, but from the new viewpoint it can no longer be regarded (at least formally) as a NR source of purely electric type. Nevertheless, it still makes sense to define NR sources based on their *leading* moment, which preserves the above classification.

VIII. CONCLUSION

This paper makes an attempt to emphasize the physical significance of a relatively unknown class of terms in the complete multipole expansion, named as MSRs. We have identified the charge-current configurations that represent MSRs and attempted to give a general recipe for their geometric interpretation, presenting several concrete examples. The fate of MSRs in classical electrodynamics is likely to be similar to that of toroidal multipoles, which themselves came to the scene only recently (although they have been known theoretically for a long time). There are two main reasons for that. First, just as toroidal multipoles, MSRs represent higher-order terms in the multipole expansion and can be neglected in most situations. Second, the radiation pattern of a MSR resembles the radiation pattern of its parent multipole, just as the radiation pattern of a toroidal multipole is indistinguishable from that of the corresponding charge multipole. Hence the presence of the MSR cannot be revealed by studying its radiation only but requires an investigation of the source as well. On the other hand, this means that the notion of MSRs is as important as that of toroidal multipoles, and the current trend shows an increasing level of interest attracted by the latter [11–20].

[5]This possibility was recently investigated in high-index dielectric particles [9,10].

ACKNOWLEDGMENTS

The authors are grateful to N. Zheludev for helpful comments. This work was partly supported by the Ministry for Education and Science of the Russian Federation in the framework of the Increase Competitiveness Program of the National University of Science and Technology MISiS under Contract No. K2-2016-051 and the Russian Foundation for Basic Research (Grants Agreement No. 16-32-50139 and No. 16-02-00789). The work on the multipoles decomposition investigation of the metamolecules was supported by Russian Science Foundation (Project No. 17-19-01786). The work of N.A.N. was partly funded by DFG Projects No. CRC/TRR 191 and No. CRC/TR 183.

APPENDIX A: COMPLETE MULTIPOLE EXPANSION

Here, for the sake of completeness, we present complete formulas expressing multipole moments and their MSRs via the charge-current density following conventions of [3]:

$$Q_{lm}^{(n)} = C_l^n \sqrt{\frac{4\pi}{2l+1}} \int d\boldsymbol{r}\, r^{l+2n} Y_{lm}^*(\hat{\boldsymbol{r}})\rho(\boldsymbol{r}), \tag{A1}$$

$$M_{lm}^{(n)} = \frac{C_l^n}{ic} \sqrt{\frac{4\pi l}{(l+1)(2l+1)}} \int d\boldsymbol{r}\, r^{2n+l} \boldsymbol{Y}_{llm}^*(\hat{\boldsymbol{r}}) \boldsymbol{j}(\boldsymbol{r}), \tag{A2}$$

$$\begin{aligned} T_{lm}^{(n)} = {}& \frac{-C_l^n}{c(2l+1)} \sqrt{\frac{4\pi l}{l+1}} \\ & \times \int d\boldsymbol{r}\, r^{2n+l+1} \left(\frac{\sqrt{l}}{2l+2n+3} \boldsymbol{Y}_{ll+1m}^*(\hat{\boldsymbol{r}}) \right. \\ & \left. + \frac{\sqrt{l+1}}{2n+2} \boldsymbol{Y}_{ll-1m}^*(\hat{\boldsymbol{r}}) \right) \boldsymbol{j}(\boldsymbol{r}). \end{aligned} \tag{A3}$$

Here $C_l^n = \frac{2^{-n}(2l+1)!!}{(2l+2n+1)!!}$ is a common combinatorial factor for all multipole families, and Y_{lm} and $\boldsymbol{Y}_{ll'm}$ are standard scalar and vector spherical harmonics, respectively [3]. In definitions of magnetic and toroidal multipoles the scalar product between vector harmonics and current density is implied. The origin of these formulas is clearly explained in Ref. [2].

APPENDIX B: FIRST-ORDER MEAN-SQUARE RADIUS OF MAGNETIC DIPOLE

Let us parametrize a torus of outer radius R and inner radius ρ by two angles ϕ and θ, see Fig. 10. Then, the current distributed on the surface of the torus and circulating parallel to the torus' equator, i.e., along the direction of vector $\hat{\boldsymbol{\phi}}$ at every point (toroidal current) has the following density:

$$\boldsymbol{j}(\boldsymbol{r}) = j_0 \int d\phi\, d\theta\, \hat{\boldsymbol{\phi}}\, \delta(\boldsymbol{r} - \boldsymbol{r}_{\phi,\theta}), \tag{B1}$$

where

$$\boldsymbol{r}_{\phi,\theta} = \begin{pmatrix} R \cos\phi + \rho \cos\phi \cos\theta \\ R \sin\phi + \rho \sin\phi \cos\theta \\ \rho \sin\theta \end{pmatrix}, \tag{B2}$$

in Cartesian coordinates. Normalization constant j_0 is related to the total current I as $j_0 = I\frac{\sqrt{R^2-\rho^2}}{2\pi}$. One can expand (B1)

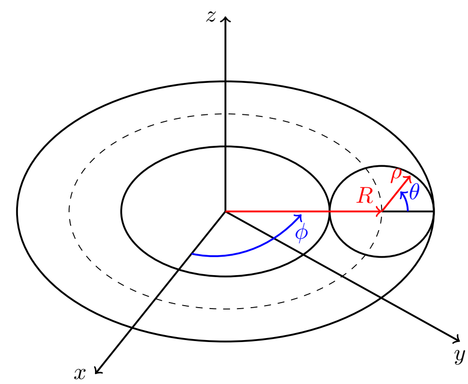


FIG. 10. Parametrization of a torus.

in the limit of large $\boldsymbol{r}$ as follows:

$$\begin{aligned} \boldsymbol{j}(\boldsymbol{r}) = j_0 \int d\phi\, d\theta\, \hat{\boldsymbol{\phi}} \bigg[& 1 - a_i \nabla_i + \frac{a_i a_j \nabla_i \nabla_j}{2} \\ & - \frac{a_i a_j a_k \nabla_i \nabla_j \nabla_k}{6} \bigg] \delta(\boldsymbol{r}) + O(r^{-4}). \end{aligned} \tag{B3}$$

Here, for brevity, we have denoted $\boldsymbol{r}_{\phi,\theta}$ as $\boldsymbol{a}$. Using the explicit Cartesian form of $\hat{\boldsymbol{\phi}} = \begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix}$ the integration in Eq. (B3) can be carried out in a straightforward manner. The result is[6]

$$\begin{aligned} \boldsymbol{j}(\boldsymbol{r}) = {}& 2\pi^2 j_0 R \,\mathrm{rot}\, \hat{z}\delta(\boldsymbol{r}) + \frac{\pi^2 j_0 R}{8} \big[(2R^2 + 3\rho^2)\big(\nabla_x^2 + \nabla_y^2\big) \\ & + 4\rho^2 \nabla_z^2 \big] \,\mathrm{rot}\, \hat{z}\delta(\boldsymbol{r}) + O(r^{-4}). \end{aligned} \tag{B4}$$

The leading term shows that the principal moment of this current configuration is the magnetic dipole moment, which is directed along the z axis and has the magnitude $2\pi^2 j_0 R$.

The expansion of a current loop can be obtained by setting $\rho = 0$ in Eq. (B4),

$$\begin{aligned} \boldsymbol{j}'(\boldsymbol{r}) = {}& 2\pi^2 j_0' R' \,\mathrm{rot}\, \hat{z}\delta(\boldsymbol{r}) \\ & + \frac{\pi^2 j_0' R'}{8} \big[2R'^2 \big(\nabla_x^2 + \nabla_y^2\big) \big] \,\mathrm{rot}\, \hat{z}\delta(\boldsymbol{r}) + O(r^{-4}), \end{aligned} \tag{B5}$$

where we have used a different notation for radius R' and total current j_0'. Assuming that

$$j_0 R = j_0' R', \tag{B6}$$

$$R' = \sqrt{R^2 - \frac{\rho^2}{2}}, \tag{B7}$$

one gets

$$\boldsymbol{j}(\boldsymbol{r}) - \boldsymbol{j}'(\boldsymbol{r}) = \frac{\pi^2}{2} j_0 \rho^2 R \,\mathrm{rot}\, \hat{z}\, \Delta\delta(\boldsymbol{r}) + O(r^{-4}) \tag{B8}$$

for the loop current placed inside the torus and circulating in the direction opposite to the toroidal current.

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[6] In Cartesian coordinates $\mathrm{rot}\, \hat{z}\delta(\boldsymbol{r}) = \begin{pmatrix} \nabla_y \\ -\nabla_x \\ 0 \end{pmatrix} \delta(\boldsymbol{r})$.

Thus, the leading moment of the resulting current configuration is the first MSR of a magnetic dipole. Condition (B6) simply ensures that the magnetic dipole moments of the torus and the loop are the same and cancel each other out. Condition (B7) ensures a more delicate balance. It defines the geometry needed for the rivals of the first MSR to vanish (see Table I).

APPENDIX C: FIRST-ORDER MEAN-SQUARE RADIUS OF TOROIDAL DIPOLE

We use the same parametrization as in the previous section to describe the surface density of the poloidal current (i.e., current that flows along the meridians of the torus),[7]

$$\boldsymbol{j}(\boldsymbol{r}) = -j_0 \int d\phi\, d\theta\, \hat{\boldsymbol{\theta}}\delta(\boldsymbol{r} - \boldsymbol{r}_{\phi,\theta}), \tag{C1}$$

with $\hat{\boldsymbol{\theta}} = \begin{pmatrix} -\sin\theta\cos\phi \\ -\sin\theta\sin\phi \\ \cos\theta \end{pmatrix}$. Expanding this current density to the order $O(r^{-5})$ yields[8]

$$\boldsymbol{j}(\boldsymbol{r}) = \pi^2 j_0 R\rho\ \mathrm{rot}^2\, \hat{\boldsymbol{z}}\delta(\boldsymbol{r}) + \frac{\pi^2}{32} j_0 R\rho\big[(4R^2+3\rho^2)\big(\nabla_x^2+\nabla_y^2\big) + 4\rho^2\nabla_z^2\big]\,\mathrm{rot}^2\, \hat{\boldsymbol{z}}\delta(\boldsymbol{r}) + O(r^{-5}). \tag{C2}$$

The leading-order term corresponds to a toroidal dipole moment directed along the z axis and having the magnitude of $c^{-1}\pi^2 j_0 R\rho = IV/4\pi c$ with V being the volume of the torus. Embedding this torus in a larger one with poloidal current $\boldsymbol{j}'$ parameters R', ρ', j_0' such that

$$j_0 R\rho = j_0' R'\rho', \tag{C3}$$

$$R'^2 - \frac{\rho'^2}{4} = R^2 - \frac{\rho^2}{4}, \tag{C4}$$

one discovers

$$\boldsymbol{j}(\boldsymbol{r}) - \boldsymbol{j}'(\boldsymbol{r}) = \frac{\pi^2}{2} j_0 R\rho(R^2 - R'^2)\Delta\ \mathrm{rot}^2\, \hat{\boldsymbol{z}}\delta(\boldsymbol{r}) + O(r^{-5}). \tag{C5}$$

Hence, the obtained current configuration corresponds to the first MSR of a toroidal dipole to the leading order. Condition (C3) ensures that the toroidal dipole moments of the two tori are the same and cancel each other out. Condition (C4) constrains the geometry of the current configuration, making sure (as in the previous case) that the first MSR is the leading term of the multipole expansion here.

[7] The normalization constant j_0 has a different relation to the total current I in this case: $j_0 = I\rho/2\pi$, the minus sign is related to the choice of direction for θ.

[8] In Cartesian coordinates $\mathrm{rot}^2\, z\delta(\boldsymbol{r}) = \begin{pmatrix} \nabla_x\nabla_z \\ \nabla_y\nabla_z \\ -\nabla_x^2-\nabla_y^2 \end{pmatrix}\delta(\boldsymbol{r})$.

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